Forcing With Elementary Substructures

(Side Condition Forcing)

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Issac Newton: "If I have seen further, it is by standing on the shoulders of giants"

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Using Forcing by Side Conditions

$$\langle V, \in, \trianglelefteq, \ldots \rangle \models \text{ZFC}$$

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Definition

$M \prec V$ if and only if

1. $M \subseteq V$
2. $\forall \varphi(v_1, \ldots, v_n) \forall a_1, \ldots, a_n \in M$

$$M \models \varphi(a_1, \ldots, a_n) \iff V \models \varphi(a_1, \ldots, a_n)$$

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Lemma

Let $V \models \mathrm{ZFC}$ *and* $\kappa \leq |V|$ *infinite cardinal*
$\forall X \subseteq V$ *with* $|X| \leq \kappa$ $\exists M \prec V$ *s.t.*

1. $|M| = \kappa$
2. $X \subseteq M$

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Some Notations

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$[x]^\kappa = \{y \subseteq x \colon |y| = \kappa\}$

IV

$\mathcal{E}_{\aleph_0,\theta} = \{M \in [H_\theta]^{\aleph_0} : M \prec H_\theta\}$

club

$C \subseteq [A]^\kappa$ is **closed unbounded** iff

1. $\forall x \in [A]^\kappa \; \exists y \in C (y \supseteq x)$
2. $\bigcup_{\alpha<\kappa} x_\alpha \in C$, for every $\subseteq$**-increasing** $\langle x_\alpha \in C \colon \alpha < \kappa \rangle$

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$C = \{\alpha \in \omega_2 \colon \alpha$ is a **limit** ordinal$\} \subseteq \omega_2$

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Definition

$S \subseteq [A]^\kappa$ is **Stationary**, if S **meets** every **club** $C \subseteq [A]^\kappa$

$S = \{\alpha \in \omega_2 \colon \mathrm{cof}(\alpha) = \omega_1\}$

$\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle \in V$ is a **Forcing** notion

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$F \subseteq \mathbb{P}$ is **filter** if

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- **predense** iff $\forall p \in \mathbb{P} \exists q \in D (q \parallel p)$

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Definition

$G \subseteq \mathbb{P}$ is **Generic** over V, if

1. G is a filter
2. $G \cap D \neq \emptyset \; \forall$ Dense/Predense $D \in V$

$\mathbb{P}$-name

$\underset{\sim}{x} = \{\langle \underset{\sim}{y}, p \rangle : p \in \mathbb{P}$ and $\underset{\sim}{y}$ is a $\mathbb{P}$-name$\}$

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Lemma

Let

1. $\check{x} = \{\langle \check{y}, 1\rangle : y \in x\}$ for every $x \in V$
2. $\dot{G} = \{\langle \check{p}, p\rangle : p \in \mathbb{P}\}$

Then for every $H \subseteq \mathbb{P}$ Generic, $\check{x}[H] = x$ and $\dot{G}[H] = H$

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Theorem

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Cohen Forcing

$\mathrm{Add}(\aleph_0, \aleph_2) = \{p \colon \omega \times \omega_2 \to 2 \colon |p| < \aleph_0\}$

1. $\cup G \colon \omega \times \omega_2 \longrightarrow 2$ **NEW!**
2. $V[G] \models \neg\mathrm{CH}$

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$A \subseteq \mathbb{P}$ is **antichain**, if $p \not\parallel q$ for all $p, q \in A$

$\mathbb{P}$ **preserves** $\theta \geq \kappa$ if its **maximal** antichains have **size** $< \kappa$

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Master Condition

$q \in \mathbb{P} \in M \in \mathcal{E}_{\aleph_0,\theta}$ is called $(M, \mathbb{P})$-**Generic condition**, if for every $D \in M$, **dense** in $\mathbb{P}$, $D \cap M$ is **predense** below q

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Theorem

$\mathbb{P}$ is proper if and only if
*For **every** large enough cardinal θ and **club many** $M \in \mathcal{E}_{\aleph_0,\theta}$*
*Every $p \in \mathbb{P} \cap M$ has an $(M,\mathbb{P})$-generic **extension***

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strong properness

Using **strongly** $(M, \mathbb{P})$-generic conditions, $D \subseteq \mathbb{P} \cap M$ must be predense below q.

The Gitik's question

Is there a **cardinal** and GCH **preserving** extension of the universe in which there is $A \subseteq \kappa$ of **size** κ such that for every **countable infinite** $x \in \mathcal{P}^V(\kappa)$

- $x \cap A \neq \emptyset$
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Let $\mathbb{P}_\kappa = \{p \colon \kappa \longrightarrow 2 \colon |p| < \aleph_0\}$

$$A = \{\alpha \colon \exists p \in G(p(\alpha) = 1)\}$$

For every $x \in \mathcal{P}(\kappa) \cap V$, let

$$D_x = \{q \in \mathbb{P} \colon \exists \gamma, \gamma' \in x(q(\gamma) = 1 \text{ and } q(\gamma') = 0)\}$$

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The problem

$\mathbb{P}_\kappa \cong \mathrm{Add}(\aleph_0, \kappa)$ So for $\kappa \geq \aleph_2$, the GCH **FAILS**!

The answer for $\aleph_2$

Let $x \subseteq \mathcal{E}_{\aleph_0,\theta}$

- $\delta_M = M \cap \omega_1$ for each $M \in x$
- $x(\alpha) = \{M \in x \colon \delta_M = \alpha\}$
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$p = \langle \mathcal{M}_p, f_p \rangle \in \mathbb{P}$ where:

1. $f_p \in \mathbb{P}_{\aleph_2}$
2. $\mathcal{M}_p \subseteq \mathcal{E}_{\aleph_0,\aleph_2}$ finite
3. $M, M' \in \mathcal{M}_p$ and $\delta_M = \delta_{M'} \implies M \cong M'$
4. $M, M' \in \mathcal{M}_p$ and $\delta_M < \delta_{M'} \implies \exists M'' \cong M'(M \in M'')$
5. If $M \in \mathcal{M}_p$, then for all $M' \in \mathcal{M}_p$ with $M \cong M'$,

$$\alpha \in \mathrm{dom}(f_p) \cap M \implies \begin{cases} \varphi_{M,M'}(\alpha) \in \mathrm{dom}(f_p) \\ f_p(\varphi_{M,M'}(\alpha)) = f_p(\alpha) \end{cases}$$

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Lemma (Strongly $(N, \mathbb{P})$-Generic Extension)

Let $\theta > \omega_2$ *large enough and* $N \in \mathcal{E}_{\aleph_0,\theta}$
Let $p \in \mathbb{P} \cap N$ *and assume* $M = N \cap H_{\omega_2}$

1. $M \in \mathcal{E}_{\aleph_0,\omega_2}$
2. $p' = \langle \mathcal{M}_p \cup \{M\}, f_p \rangle$ *is the master condition*

Strongly Master Condition

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Lemma

By ***Elementarity****, there exists* $\hat{q} = \langle \hat{\mathcal{M}}_q, \hat{f}_q \rangle \in \mathbb{P} \cap N$

1. $\hat{q} \parallel q$
2. $\mathrm{supp}(\hat{\mathcal{M}}_q) = \mathrm{supp}(\mathcal{M}_q) \cap N$
3. $\mathcal{M}_q \cap N \subseteq \hat{\mathcal{M}}_q$
4. $M_1 \in \mathcal{M}_q$ *and* $M_2 \in \hat{\mathcal{M}}_q(\delta_{M_1}) \implies M_1 \cong M_2$
5. $\hat{f}_q \supseteq f_q \restriction N$

The restriction of q to N

Let W be the set of all $\in$-chains $w = \langle N_1^w, \dots, N_l^w \rangle$ in $\hat{\mathcal{M}}_q \cup \mathcal{M}_q$ such that $N_l^w \in \mathcal{M}_q(\delta_M)$

$q \restriction N = \langle \mathcal{M}_q \restriction M, f_q \restriction M \rangle$ where

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Lemma

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Let $r \in D$, **extends** $q \restriction N$

$s = \langle \mathcal{M}_s, f_s \rangle$ where
$\mathcal{M}_s = \mathcal{M}_r \cup \mathcal{M}_q \cup \{\varphi_{M,N}(K) : N \in \mathcal{M}_q(\delta_M) \text{ and } K \in \mathcal{M}_r\}$
$f_s = f_r \cup f_q \cup \{\langle \varphi_{N',N''}(\alpha), f_r(\alpha) \rangle :$
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(GCH) $\mathbb{P}$ is $\aleph_2-$c.c so preserves **all** cardinals

Lemma

Let $p \in M \cong M' \in \mathcal{E}_{\aleph_0,\aleph_2}$,

1. $p_{M,M'} = \langle \mathcal{M}_p \cup \{M, M'\}, f_p \cup \{\langle \varphi_{M,M'}(\alpha), f_p(\alpha)\rangle : \alpha \in \mathrm{dom}(f_p)\}\rangle$
2. $p_{M,M'} \Vdash \check{\varphi}_{M,M'}[\dot{G} \cap \check{M}] = \dot{G} \cap \check{M'}$

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Lemma

$\mathbb{P}$ *preserves the* CH

By **contradiction** assume $p \Vdash \langle \underset{\sim}{r}_\alpha : \alpha < \omega_2\rangle$ is a set of **reals**
Let $p_\alpha \leq p$ and $p_\alpha \Vdash \underset{\sim}{r}_\alpha \subseteq \check{\omega}$
Let $p_\alpha, p, \mathbb{P}, \underset{\sim}{r}_\alpha \in M_\alpha \in \mathcal{E}_{\aleph_0,\theta}$
There is $\alpha < \beta < \omega_2$ s.t. $\langle M_\alpha, \in, \mathbb{P}, p_\alpha, \underset{\sim}{r}_\alpha\rangle \cong \langle M_\beta, \in, \mathbb{P}, p_\beta, \underset{\sim}{r}_\beta\rangle$
$p_{M_\alpha,M_\beta} \leq p_\alpha, p_\beta$ and $p_{M_\alpha,M_\beta} \Vdash \underset{\sim}{r}_\alpha = \underset{\sim}{r}_\beta$

Theorem

*Let $T \subseteq \kappa$ **stationary**. There are **stationary** sets $S_1, S_2 \subseteq T$ such that $S_1 \cap S_2 = \emptyset$*

Adding a club

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Baumgartner, Harrington and Kleinberg (1976)

Let $T \subseteq \omega_1$ **stationary**. There is a forcing notion which adds a **club** $C \subseteq T$

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(1983)Abraham-Shelah

Let $S \subseteq \kappa$ **fat stationary**. There is a forcing notion which adds a **club** $C \subseteq S$ and adds **no new** sets of **size** $< \kappa$

Still adding club

2004 Mitchell, 2005 Friedman

A **club** in ω_2 with **finite** conditions
The first use of **side conditions** to add an object to a cardinal $> \aleph_1$

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2014 Neeman

A **club** in ω_2 with **finite** condition contains $\in$-chains of **elementary substructures** of **two** types

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2004 Mitchell, 2005 Friedman

A **club** in ω_2 with **finite** conditions
The first use of **side conditions** to add an object to a cardinal $> \aleph_1$

2014 Neeman

A **club** in ω_2 with **finite** condition contains $\in$-chains of **elementary substructures** of **two** types

We (2024)?

A Generalization of Abraham's Forcing from another aspect

$\mathcal{N} = \langle N_\xi : \xi \leq \alpha \rangle$ is an α-**tower** where

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α is **indecomposable**, if $\beta + \gamma < \alpha$, for **every** $\beta \leq \gamma < \alpha$

Definition

$p = \langle \mathcal{M}_p, f_p \rangle$ where

1. $\mathcal{M}_p = \langle M_i^p \in \mathcal{E}_{\aleph_0,\aleph_1} : i < n_p \rangle$ **finite** $\in$-chain
2. $f_p : \mathcal{M}_p \longrightarrow H_{\omega_1}$ such that

- $f_p(M_i^p) \subset M_{i+1}^p$ **finite**
- $f_p(M_{n_p-1}^p) \subset H_{\omega_1}$ **finite**

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$q \leq p$ if and only if

1. $\mathcal{M}_p \subseteq \mathcal{M}_q$
2. $f_p(M) \subseteq f_q(M)$, for **all** $M \in \mathcal{M}_p$

$\mathbb{P}$ Adds a club

Theorem

$C = \{\delta_M \colon \exists p \in G(M \in \mathcal{M}_p)\}$ *is a* ***club*** *in* ω_1

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Lemma

$D_\gamma = \{q \in \mathbb{P} \colon \exists M \in \mathcal{M}_q(\gamma < \delta_M)\}$ *is* ***Dense***

$(\forall p \in \mathbb{P})(\forall \gamma \in \omega_1)(\exists p' \leq p)(\exists N \in \mathcal{M}_{p'})(\gamma < \delta_N)$

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Lemma

If $p \Vdash \check{\gamma} \notin \dot{C}$, *then* $p \Vdash \check{\gamma} \notin \lim \dot{C}$

$$\delta_{M_i^{p'}} < \gamma < \delta_{M_{i+1}^{p'}} \implies \gamma < \xi \in \delta_{M_{i+1}^{p'}}$$
$$f_q(M_i^{p'}) = f_{p'}(M_i^{p'}) \cup \{\xi\}$$
$$r \leq q \implies r \Vdash \dot{C} \cap (\delta_{M_i^{p'}}, \xi] = \emptyset$$

Theorem

$\mathbb{P}$ *is proper*

Let $p \in M' \in \mathcal{E}_{\aleph_0,\theta}$ and $M = M' \cap H_{\omega_1}$
$p' = \langle \mathcal{M}_p \cup \{M\}, f_p \cup \langle M, \emptyset \rangle \rangle$
Let $D \subseteq \mathbb{P} \cap M'$ and $q \leq p'$
$q{\restriction}M' = \langle \mathcal{M}_q \cap M, f_q{\restriction}M \rangle$
Let $r \leq q{\restriction}M'$
$s = \langle \mathcal{M}_r \cup \mathcal{M}_q, f_r \cup f_q \rangle$

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Theorem

$\mathbb{P}$ *is* ***not*** ω*-proper*

Let $\mathcal{N} = \langle N_i \colon i \leq \omega \rangle$ be an ω-tower and $p \in N_0$
If $q \leq p$ is $(\mathcal{N}, \mathbb{P})$-generic, then $q \Vdash \dot{C}$ is **not** a club

Definition

Let $\alpha < \omega_1$ **indecomposable**

$\mathbb{P}[\alpha] = \{p = \langle \mathcal{M}_p, f_p, \mathcal{W}_p \rangle\}$ where

- $\mathcal{M}_p = \langle M^p_\xi \in \mathcal{E}_{\aleph_0,\aleph_1} : \xi \leq \gamma_p < \alpha \rangle$ a **continuous** $\in$-chain
- $f_p : \mathcal{M}_p \longrightarrow H_{\omega_1}$ s.t.
  $f_p(M^p_\xi) \subset M^p_{\xi+1}$ **finite**
  $f_p(M^p_\gamma) \subset H_{\omega_1}$ **finite**
- $\mathcal{W}_p \subset \mathcal{M}_p$ s.t. $\forall N \in \mathcal{W}_p, p{\restriction}N = \langle \mathcal{M}_p \cap N, f_p{\restriction}N, \mathcal{W}_p \cap N \rangle \in N$

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3. $\mathcal{W}_p \subseteq \mathcal{W}_q$

Generalized case

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$C = \{\delta_M : \exists p \in G(M \in \mathcal{M}_p)\}$ is a **club** in ω_1

Theorem

$\mathbb{P}[\alpha]$ ***is*** $<\alpha$*-proper but* ***not*** α*-proper*

Theorem

$\mathbb{P}[\alpha]$ ***is*** $< \alpha$*-proper but* ***not*** α*-proper*

$\mathcal{N} = \langle N_\zeta : \zeta \leq \beta < \alpha \rangle$ and $p \in \mathbb{P}[\alpha] \cap N_0$ fix
$p' = \langle \mathcal{M}_{p'}, f_{p'}, \mathcal{W}_{p'} \rangle$ such that

1. $\mathcal{M}_{p'} = \mathcal{M}_p \cup \mathcal{N}$
2. $f_{p'}(M_\xi^{p'}) = f_p(M_\xi^p)$ for $\xi \leq \gamma_p$
3. $f_{p'}(N_\zeta) = \delta_{N_\zeta} + 1$ for $\zeta \leq \beta$
4. $\mathcal{W}_{p'} = \mathcal{W}_p \cup \{N_\zeta \in \mathcal{N} : \zeta$ is **not** a **limit**$\}$

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p' is **strongly** $(N_\zeta, \mathbb{P}[\alpha])$-generic if ζ is **non-limit**
p' is $(N_\delta, \mathbb{P}[\alpha])$-**generic** if δ is **limit**

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Let $\mathcal{N} = \langle N_\xi : \xi \leq \alpha \rangle$ is an α-tower and $p \in N_0$
If $q \leq p$ is $(\mathcal{N}, \mathbb{P}[\alpha])$-generic, then $q \Vdash \dot{C}$ is **not** a **club**

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Thank You