Two side conditions forcing preserving the GCH

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$\aleph_1$-PRESERVING FORCINGS

Proper Forcings

A forcing notion $\mathbb{P}$ is proper if for every infinite X and every stationary set $S \subseteq [X]^{\leq \aleph_0}$, S remains stationary in $V[G]$

- **Shelah:** If $\mathbb{P}$ is proper then forcing with $\mathbb{P}$ preserves $\aleph_1$
- **Shelah:** Countable support iteration of proper forcing notions is proper

PROPERNESS & STRONGLY PROPERNESS

Let θ be a large enough regular cardinal, $\mathbb{P}$ a forcing notion and $\langle N, \in, \leq_w \rangle \prec \langle H(\theta), \in, \leq_w \rangle$ countable with $\mathbb{P} \in N$

q is $(N, \mathbb{P})$-generic condition if for every dense open set $D \subset \mathbb{P}$, if $D \in N$ then $D \cap N$ is predense below q

$\mathbb{P}$ is proper iff for all such N, every $p \in \mathbb{P} \cap N$ has an $(N, \mathbb{P})$-generic extension

$q \in \mathbb{P}$ is $(N, \mathbb{P})$-strongly generic condition if every dense open set $D \subseteq \mathbb{P} \cap N$ is predense below q

$\mathbb{P}$ is strongly proper iff for all such N, every $p \in \mathbb{P} \cap N$ has an $(N, \mathbb{P})$-strongly generic extension

THE $\in$-COLLAPSE FORCING

Todorcevic 1984

$\mathbb{P}_{\in}(\theta)$ is the set of all finite $\in$-chains of countable elementary submodels of $\langle H(\theta), \in, \leq_w \rangle$ with the inverse inclusion

Let $\kappa > \theta$ be a large enough regular cardinal and $\langle M', \in, <_w \rangle \prec \langle H(\kappa), \in, <_w \rangle$ with $\theta \in M'$ and $M = M' \cap H(\theta)$

- If $M \in q$, then $q \cap M' \in \mathbb{P}_{\in}(\theta) \cap M'$
- If $p \in \mathbb{P}_{\in}(\theta) \cap M'$, then $p \cup \{M\} \in \mathbb{P}_{\in}(\theta)$.

Applications

PFA implies *OGA*, *PID*, $BA(\omega_1)$

MATRIX $\in$-COLLAPSE FORCING

Todorcevic 2017

$$\mathcal{S} = \{M \in [H(\theta)]^{\aleph_0} : \langle M, \in, <_w\rangle \prec \langle H(\theta), \in, <_w\rangle\}$$

$\mathbb{P}_{\in}^{\mathcal{M}} = \{p \subset \mathcal{S} \mid |p| < \aleph_0\}$ such that

- If $M, N \in p$ and $M \cap \omega_1 = \delta_M = \delta_N = N \cap \omega_1$, then

  $\langle M, \in, <_w\rangle \simeq \langle N, \in, <_w\rangle$

- If $M \in p$ and $\delta \in dom(p)$ such that $\delta_M < \delta$, then

  $\exists N \in p(\delta)\ (M \in N)$.

  where $p(\alpha) = \{M \in p : \delta_M = \alpha\}$ and $dom(p) = \{\alpha : p(\alpha) \neq \emptyset\}$.

$\mathbb{P}_{\in}^{\mathcal{M}}$ is strongly proper and forces the Continuum Hypothesis

Aplications

- In $V[G]$ there is a Kurepa tree with exactly ω_2 branches that does not contain Aronszajn subtrees

Gitik's question (2017)

Let $\kappa \geq \aleph_2$ be a regular cardinal. Is there a cardinal and GCH preserving extension in which there exists a set $A \subseteq \kappa$ of size κ such that for all countable set $X \in \mathscr{P}(\kappa) \cap V$, $A \cap X$ and $X \backslash A$ are non-empty?

For all κ, $\mathbb{P}_\kappa = \{p : \kappa \longrightarrow 2 : |p| < \aleph_0\} \cong Add(\omega, \kappa)$ forces the existens of such set A, but it also forces $2^{\aleph_0} \geq \kappa$. So for $\kappa \geq \aleph_2$ the CH fails

Theorem (R.Hoseininaveh-M.Golshani)

The answer to Gitik's question is Yes for $\kappa = \aleph_2$.

Definition

Let $p = \langle \mathcal{M}_p, f_p \rangle \in \mathbb{P}$ iff

- $\mathcal{M}_p \in \mathbb{P}^M_\in$
- $f_p : \omega_2 \longrightarrow 2$ is a finite partial function
- Suppose $M, N \in \mathcal{M}_p$ and $\delta_M = \delta_N$. for all $\alpha < \omega_2$, if $\alpha \in \mathrm{dom}(f_p) \cap M$, then $\varphi_{M,N}(\alpha) \in \mathrm{dom}(f_p)$ and $f_p(\varphi_{M,N}(\alpha)) = f_p(\alpha)$

Theorem

$\mathbb{P}$ *is strongly proper*

Theorem

$\mathbb{P}$ *has* $\aleph_2$*.c.c*

Theorem

$\mathbb{P}$ *preserves the CH*

Let $X \in \mathcal{P}(\omega_2) \cap V$ be countable.

$D_X = \{p \in \mathbb{P} : \exists \alpha, \beta \in \mathrm{dom}(f_p) \cap X \ s.t.\ f_p(\alpha) = 1 \wedge f_p(\beta) = 0\}$ is a dense subset of $\mathbb{P}$.

$\kappa < \lambda < \theta$

- $\mathcal{T}$ is a collection of transitive $\langle W, \in, <_w \rangle \prec \langle H(\theta), \in, <_w \rangle$, and $\mathcal{S}$ is a collection of $\langle M, \in, <_w \rangle \prec \langle H(\theta), \in, <_w \rangle$ with $\kappa \subset M$ and $|M| < \lambda$. All elements of $\mathcal{S} \cup \mathcal{T}$ belong to $H(\theta)$ and contain $\{\kappa, \lambda\}$
- If $M_1, M_2 \in \mathcal{S}$ and $M_1 \in M_2$ then $M_1 \subset M_2$
- If $M \in \mathcal{S}$ and $W \in M \cap \mathcal{T}$ then $(M \cap W) \in W \cap \mathcal{S}$
- Each $W \in \mathcal{T}$ is closed under sequences of lenght $\leq \kappa$ in $H(\theta)$

$\mathbb{P}^{\mathcal{S},\mathcal{T}}_{\kappa,\lambda} = \{\langle M_\xi : \xi < \gamma\rangle \in H(\theta) \mid \gamma < \kappa\}$ such that

- $\forall \xi,\ M_\xi \in \mathcal{S} \cup \mathcal{T}$
- $\forall \zeta < \gamma,\ \{\xi < \zeta : M_\xi \in M_\zeta\}$ is cofinal in ζ
- $\forall \zeta < \gamma,\ \langle M_\xi : \xi < \zeta \wedge M_\xi \in M_\zeta\rangle \in M_\zeta$
- The sequence is closed under intersections

TWO APPLICATIONS

$$\mathcal{S} = \{M \in [H(\theta)]^{\aleph_0} : \langle M, \in, <_w \rangle \prec \langle H(\theta), \in, <_w \rangle\}$$

$$\mathcal{T} = \{H(\lambda) : \langle H(\lambda), \in, <_w \rangle \prec \langle H(\theta), \in, <_w \rangle \wedge cf(\lambda) > \omega\}$$

1

$\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$ preserves $\aleph_1$

If λ be a cardinal such that $\omega_1 < \lambda < \theta$, then λ is collapsed to ω_1

If $\mathcal{T}$ is a stationary set on $[H(\theta)]^{<\theta}$ then $\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}} \Vdash \theta = \omega_2$

2

Let θ a Supercompact cardinal and let $J : \theta \longrightarrow H(\theta)$ be the Laver function

$\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}(J)$ is an iterated forcing, constructed using $\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$. It satisfies:

(1) preserves $\aleph_1$

(2) forces that $\theta = \aleph_2$

(3) Gives a new proof for the consistency of PFA

(4) $V[G]$ is a model of PFA in which PFA^+ fails

THE QUESTION

The Problem

The forcing of Neeman does not preserve GCH. Can we define a version of it preserving the GCH?

The Idea

Replacing the linear part of small nodes with matrices of the same types of models

Notations

Let $<_w$ be a well-ordering of $H(\omega_2)$. We will fix the following two families:

$$\mathcal{S} = \{M \in [H(\omega_2)]^{\aleph_0} : \langle M, \in, <_w\rangle \prec \langle H(\omega_2), \in, <_w\rangle\}$$

$$\mathcal{T} = \{X \in [H(\omega_2)]^{\aleph_1} : \langle X, \in, <_w\rangle \prec \langle H(\omega_2), \in, <_w\rangle \wedge {}^{\omega}X \subset X\}$$

Notations

We use:

- M, N, L and K for elements of $\mathcal{S}$ which are called small nodes
- W, X, Y and Z for elements of $\mathcal{T}$ which are called transitive nodes
- A, B, C and D when the types are not important
- Let $\langle A, \in, <_w\rangle$ and $\langle B, \in, <_w\rangle$ be isomorphic. We denote this unique isomorphism by $\varphi_{A,B} : A \longrightarrow B$

Notations

- For $A, B \in \mathcal{S} \cup \mathcal{T}$ with $A \ncong B$ we say that $A \in^\star B$, if there is an $\in$-path from A to B; i.e. there are $C_0, C_1, \ldots, C_n \in \mathcal{S} \cup \mathcal{T}$ such that $C_i \in C_{i+1}$ for all $i < n$ and $A = C_0, B = C_n$.
- Given $A, B \in p \subset \mathcal{S} \cup \mathcal{T}$, $(A, B)_p = \{C \in p : A \in^\star C \in^\star B\}$
- Given $A \in p \subset \mathcal{S} \cup \mathcal{T}$, $p_{<A} = \{B \in p \ : \ B \in^\star A\}$

$\mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}$ is the set of all finite $p \subset \mathcal{S} \cup \mathcal{T}$ such that the following conditions hold.

$(P1)$ If $A, B \in p$ and $A \ncong B$, then either:

(i) there exists $B' \in p$ such that $B \cong B'$ and $A \in^\star B'$ in p; or

(ii) there exists $A' \in p$ such that $A \cong A'$ and $B \in^\star A'$ in p;

$(P2)$ If $M, N \in p \cap \mathcal{S}, \delta_M = \delta_N$ and $p_{<M} \cap \mathcal{T} = p_{<N} \cap \mathcal{T}$, then $M \cong N$. Furthermore for every $X \in (M \cap N) \cap \mathcal{T}$, $\langle M, \in, <_w, X \cap M\rangle$ is isomorphic to $\langle N, \in, <_w, X \cap N\rangle$;

$(P3)$ If $W, X \in p \cap \mathcal{T}$ and $\delta_W = \delta_X$, then $X = W$;

$(P4)$ If $A \in^\star B$, then $A \cap B \in p$. Furtheremore if $A \in^\star B \in^\star C$, then either $A = B \cap C$ or $A \in^\star B \cap C$ or $B \cap C \in^\star A$

Lemma

For all $M \in \mathcal{S}$ *and* $X \in \mathcal{T}$, $\delta_{X \cap M} = \delta_M$

If $X \in^\star M$ *in a condition* p*, then* $X \cap M$ *can not be isomorphic to* M.

Fact

If $A \in^\star M$ *it is not necessary to have* $A \in M$

If $M \in p \in \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}$*, it is not necessarily the case that* $p \cap M$ *is a condition*

If $M \prec H(\omega_2)$ *and* $p \in \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}} \cap M$ *it is not always true that* $p \cup \{M\} \in \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}$.

Lemma

Let θ be a large enough regular cardinal, M' be a countable elementary submodel of $H(\theta)$, Let $q \in \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}$ be such that $M' \cap H(\omega_2) = M \in q$.

Then there is $\hat{q}$ such that:

(1) $\hat{q} \in \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}} \cap M$;

(2) $\mathrm{dom}(\hat{q}) = \mathrm{dom}(q \cap M)$;

(3) $q \cap M \subseteq \hat{q}$;

(4) *If* $\alpha \in \mathrm{dom}(\hat{q})$ *and* $N_1 \in q(\alpha) \cap \mathcal{S}, N_2 \in \hat{q}(\alpha) \cap \mathcal{S}$, *and* $N_1 \cap \mathcal{T} = N_2 \cap \mathcal{T}$, *then* $N_1 \cong N_2$;

(5) *If* $X_1 \in \hat{q} \cap \mathcal{T}, X_2 \in q \cap \mathcal{T}$ *and* $\delta_{X_1} = \delta_{X_2}$, *then* $X_1 = X_2$;

(6) $\hat{q} \cap \mathcal{T} = q \cap M \cap \mathcal{T}$

(7) $\hat{q} \cup q \in \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}$

Let $q \in \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}$ be such that $M' \cap H(\omega_2) = M \in q$ and let

$\mathcal{F} = \{w = \langle A^w_l \in A^w_{l-1} \in \cdots \in A^w_0 \rangle\}$ be such that

$A^w_i \in q \cup \hat{q}$, $A^w_0 \in q(\delta_M)$ with $A^w_0 \cong M$.

$q{\restriction}M = \{\varphi_{A^w_n, B_n}(A^w_{n+1}) : w \in \mathcal{F} \wedge n < l\}$ where $B_0 = M$ and $B_n \in^\star M$ for $n > 0$ is either an element of $q \cup \hat{q}$ or is of the form $\varphi_{A^{w'}_k, B_k}(A^{w'}_{k+1})$ for some $w' \in \mathcal{F}$.

Lemma

Let M' be a countable elementary submodel of $H(\theta)$, with $\mathbb{P}_{\in,\mathcal{M}}^{\mathcal{S},\mathcal{T}}, \mathcal{S}, \mathcal{T} \in M'$ for some $\theta > \omega_2$, a large enough regular cardinal. Let $q \in \mathbb{P}_{\in,\mathcal{M}}^{\mathcal{S},\mathcal{T}}$ be such that $M' \cap H(\omega_2) = M \in q$.

(1) $q_{<M} \cup \hat{q} \subseteq q\restriction M$;

(2) $X \in (q\restriction M) \cap \mathcal{T}$ *iff* X *is a transitive nodes of* q *which is before* M;

(3) $q\restriction M \in \mathbb{P}_{\in,\mathcal{M}}^{\mathcal{S},\mathcal{T}}$;

(4) $(q\restriction M) \cap M \in \mathbb{P}_{\in,\mathcal{M}}^{\mathcal{S},\mathcal{T}} \cap M$.

Lemma

Let $q \in \mathbb{P}_{\in,\mathcal{M}}^{\mathcal{S},\mathcal{T}}$ and $M \in q \cap \mathcal{S}$. If $r \leq (q\restriction M) \cap M$ and $r \in M$, then $t = (q\restriction M) \cup r \cup q$ satisfies $P1$ to $P3$.

For each $M' \in q$ with $M \cong M'$, first take $X \in r\backslash(q\restriction M) \cap \mathcal{T}$ such that $(X, M')_t \cap q\restriction M \cap \mathcal{T} \neq \emptyset$. Let X' be the smallest element of this set, then $X \in M' \cap X'$.

Define

$E_{M'}(X) = [M' \cap X', X')_{q \cup q\restriction M}$

$F_{M'}(X) = \{X \cap N : N \in E_{M'}(X)\}$

Where $X \in r\backslash(q\restriction M) \cap \mathcal{T}$ is such that $(X, M')_t \cap q\restriction M \cap \mathcal{T} = \emptyset$

Define

$E_{M'}(X)$ is the largest interval of small nodes of q starting from every $M' \cong M$.

$F_{M'}(X) = \{X \cap N : N \in E_{M'}(X)\}$

Now we define $s = t \cup \bigcup_{X \in r \setminus (q \restriction M) \cap \mathcal{T}} F_{M'}(X)$.

Lemma

$s \in \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}$ and extends q and r.

Theorem

$\mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}$ is strongly proper.

Proof.

Let θ be a large enough regular cardinal and $M' \prec H(\theta)$ countable with $p \in \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}} \cap M'$. Let $M = M' \cap H(\omega_2)$ and $p' = \{M\}$. then $p \in M$ and extends $p' \cap M$. Hence there is $q \in \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}$ such that q extends both p and p'. We can find $\hat{q}$, $q \restriction M$ as before, so suppose $r \in D \subset \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}} \cap M'$ which $r \leq (q \restriction M \cap M)$. In this way, we can find s which extends both r and q. So q is a $(\mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}, M')$-strongly generic conditin. $\square$

Theorem

Forcing with $\mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}$ *preserves the GCH.*

Thank You