Adding Abraham clubs and α-properness

Rouholah Hoseini Naveh
Joint with
Mohammad Golshani

Seminar on Mathematical Logic and its Applications
IPM - 1403/3/9

Some Well known Definitions

club

Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is a **closed unbounded subset (club)** in κ if:

Some Well known Definitions

club

Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is a **closed unbounded subset (club)** in κ if:

1. C is unbounded in κ

Some Well known Definitions

club

Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is a **closed unbounded subset (club)** in κ if:

1. C is unbounded in κ i.e. $\sup(C \cap \kappa) = \kappa$;

Some Well known Definitions

club

Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is a **closed unbounded subset (club)** in κ if:

1. C is unbounded in κ i.e. $\sup(C \cap \kappa) = \kappa$;
2. C contains all its limit points

Some Well known Definitions

club

Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is a **closed unbounded subset (club)** in κ if:

1. C is unbounded in κ i.e. $\sup(C \cap \kappa) = \kappa$;
2. C contains all its limit points i.e. $\forall \alpha < \kappa \sup(C \cap \alpha) \in C$.

Some Well known Definitions

club

Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is a **closed unbounded subset (club)** in κ if:

1. C is unbounded in κ i.e. $\sup(C \cap \kappa) = \kappa$;
2. C contains all its limit points i.e. $\forall \alpha < \kappa \sup(C \cap \alpha) \in C$.

Stationary

A set $S \subseteq \kappa$ is **stationary** if $S \cap C \neq \emptyset$ for every club C of κ.

Some Well known Definitions

club

Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is a **closed unbounded subset (club)** in κ if:

1. C is unbounded in κ i.e. $\sup(C \cap \kappa) = \kappa$;
2. C contains all its limit points i.e. $\forall \alpha < \kappa \sup(C \cap \alpha) \in C$.

Stationary

A set $S \subseteq \kappa$ is **stationary** if $S \cap C \neq \emptyset$ for every club C of κ.

Notation: $[X]^\kappa$

$\{Y \subseteq X : |Y| = \kappa\}$, for a cardinal $\kappa \leq |X|$

Some Well known Definitions

club

Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is a **closed unbounded subset (club)** in κ if:

1. C is unbounded in κ i.e. $\sup(C \cap \kappa) = \kappa$;
2. C contains all its limit points i.e. $\forall \alpha < \kappa \sup(C \cap \alpha) \in C$.

Stationary

A set $S \subseteq \kappa$ is **stationary** if $S \cap C \neq \emptyset$ for every club C of κ.

Notation: $[X]^\kappa$

$\{Y \subseteq X : |Y| = \kappa\}$, for a cardinal $\kappa \leq |X|$

Proper Forcing

Shelah: A notion of forcing $\mathbb{P}$ is **proper** if for every uncountable cardinal λ, every stationary subset of $[\lambda]^{\aleph_0}$ remains stationary in the generic extension.every countable set of ordinals in the extension is covered by a countable set of

H_λ

Let λ be a regular cardinal. H_λ is the collection of all sets hereditarily of cardinality less than λ,

H_λ

Let λ be a regular cardinal. H_λ is the collection of all sets hereditarily of cardinality less than λ, i.e. $\{x\colon |trcl(x)| < \lambda\}$

H_λ

Let λ be a regular cardinal. H_λ is the collection of all sets hereditarily of cardinality less than λ, i.e. $\{x\colon |trcl(x)| < \lambda\}$

Master condition

Let $M \prec H_\lambda$. A condition $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic if for every dense open $D \subseteq \mathbb{P}$ with $D \in M$, the set $D \cap M$ is predense below q,

H_λ

Let λ be a regular cardinal. H_λ is the collection of all sets hereditarily of cardinality less than λ, i.e. $\{x \colon |trcl(x)| < \lambda\}$

Master condition

Let $M \prec H_\lambda$. A condition $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic if for every dense open $D \subseteq \mathbb{P}$ with $D \in M$, the set $D \cap M$ is predense below q, i.e. $(\forall q' \leq q)(\exists r \in D \cap M)(\exists s \in \mathbb{P})(s \leq q', r)$

H_λ

Let λ be a regular cardinal. H_λ is the collection of all sets hereditarily of cardinality less than λ, i.e. $\{x\colon |trcl(x)| < \lambda\}$

Master condition

Let $M \prec H_\lambda$. A condition $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic if for every dense open $D \subseteq \mathbb{P}$ with $D \in M$, the set $D \cap M$ is predense below q, i.e. $(\forall q' \leq q)(\exists r \in D \cap M)(\exists s \in \mathbb{P})(s \leq q', r)$

Theorem (Shelah)

A forcing notion $\mathbb{P}$ is proper if and only if for every large enough regular cardinals λ, and for club many ***countable*** *$M \prec H_\lambda$, for every condition $p \in M$, there is $q \leq p$ which is an $(M, \mathbb{P})$-generic condition.*

H_λ

Let λ be a regular cardinal. H_λ is the collection of all sets hereditarily of cardinality less than λ, i.e. $\{x\colon |trcl(x)| < \lambda\}$

Master condition

Let $M \prec H_\lambda$. A condition $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic if for every dense open $D \subseteq \mathbb{P}$ with $D \in M$, the set $D \cap M$ is predense below q, i.e. $(\forall q' \leq q)(\exists r \in D \cap M)(\exists s \in \mathbb{P})(s \leq q', r)$

Theorem (Shelah)

A forcing notion $\mathbb{P}$ is proper if and only if for every large enough regular cardinals λ, and for club many ***countable*** *$M \prec H_\lambda$, for every condition $p \in M$, there is $q \leq p$ which is an $(M, \mathbb{P})$-generic condition.*

strong properness

using strongly $(M, \mathbb{P})$-generic conditions,

H_λ

Let λ be a regular cardinal. H_λ is the collection of all sets hereditarily of cardinality less than λ, i.e. $\{x\colon |trcl(x)| < \lambda\}$

Master condition

Let $M \prec H_\lambda$. A condition $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic if for every dense open $D \subseteq \mathbb{P}$ with $D \in M$, the set $D \cap M$ is predense below q, i.e. $(\forall q' \leq q)(\exists r \in D \cap M)(\exists s \in \mathbb{P})(s \leq q', r)$

Theorem (Shelah)

A forcing notion $\mathbb{P}$ is proper if and only if for every large enough regular cardinals λ, and for club many ***countable*** *$M \prec H_\lambda$, for every condition $p \in M$, there is $q \leq p$ which is an $(M, \mathbb{P})$-generic condition.*

strong properness

using strongly $(M, \mathbb{P})$-generic conditions, $D \subseteq \mathbb{P} \cap M$ must be predense below q.

preserving $\aleph_1$

If $\mathbb{P}$ is strongly proper, then $\mathbb{P}$ is proper, in particular forcing with $\mathbb{P}$ does not collapse $\aleph_1$.

preserving $\aleph_1$

If $\mathbb{P}$ is strongly proper, then $\mathbb{P}$ is proper, in particular forcing with $\mathbb{P}$ does not collapse $\aleph_1$.

Tower

Let $\alpha < \omega_1$. The sequence $\mathcal{N} = \langle N_\xi : \xi \leq \alpha \rangle$ is said to be an α-tower if for some regular cardinal λ,

1. $N_\xi \prec H_\lambda$, countable;
2. $N_\xi \in N_{\xi+1}$ for $\xi < \alpha$;
3. $N_\delta = \bigcup_{\xi<\delta} N_\xi$ for limit ordinals $\delta \leq \alpha$;
4. $\langle N_\zeta : \zeta \leq \xi \rangle \in N_{\xi+1}$ for every $\xi < \alpha$.

preserving $\aleph_1$

If $\mathbb{P}$ is strongly proper, then $\mathbb{P}$ is proper, in particular forcing with $\mathbb{P}$ does not collapse $\aleph_1$.

Tower

Let $\alpha < \omega_1$. The sequence $\mathcal{N} = \langle N_\xi : \xi \leq \alpha \rangle$ is said to be an α-tower if for some regular cardinal λ,

1. $N_\xi \prec H_\lambda$, countable;
2. $N_\xi \in N_{\xi+1}$ for $\xi < \alpha$;
3. $N_\delta = \bigcup_{\xi<\delta} N_\xi$ for limit ordinals $\delta \leq \alpha$;
4. $\langle N_\zeta : \zeta \leq \xi \rangle \in N_{\xi+1}$ for every $\xi < \alpha$.

α-properness

$\mathbb{P}$ is called α-proper if for every α-tower with $\mathbb{P} \in N_0$, every condition $p \in \mathbb{P} \cap N_0$ has an extension q which is a master condition for each $N_\xi \in \mathcal{N}$.

Notation: $\mathcal{S}$

The collection of all countable $M \prec H_{\omega_1}$.

Notation: $\mathcal{S}$

The collection of all countable $M \prec H_{\omega_1}$.

Notation: δ_M

The ordinal $M \cap \omega_1$, for $M \in \mathcal{S}$.

Notation: $\mathcal{S}$

The collection of all countable $M \prec H_{\omega_1}$.

Notation: δ_M

The ordinal $M \cap \omega_1$, for $M \in \mathcal{S}$.

Let $\mathbb{P}$ consist of pairs $p = \langle \mathcal{M}_p, f_p \rangle$, where

1. $\mathcal{M}_p = \langle M_i^p : i < n_p \rangle$ is a finite $\in$-increasing sequence of elements of $\mathcal{S}$, and

Notation: $\mathcal{S}$
The collection of all countable $M \prec H_{\omega_1}$.

Notation: δ_M
The ordinal $M \cap \omega_1$, for $M \in \mathcal{S}$.

Let $\mathbb{P}$ consist of pairs $p = \langle \mathcal{M}_p, f_p \rangle$, where

1. $\mathcal{M}_p = \langle M_i^p : i < n_p \rangle$ is a finite $\in$-increasing sequence of elements of $\mathcal{S}$, and
2. the function $f_p : \mathcal{M}_p \longrightarrow H_{\omega_1}$ is defined such that $f_p(M_i^p)$ is a finite subset of M_{i+1}^p if $i < n_p - 1$, and $f_p(M_{n_p-1}^p)$ is a finite subset of H_{ω_1}.

Notation: $\mathcal{S}$
The collection of all countable $M \prec H_{\omega_1}$.

Notation: δ_M
The ordinal $M \cap \omega_1$, for $M \in \mathcal{S}$.

Let $\mathbb{P}$ consist of pairs $p = \langle \mathcal{M}_p, f_p \rangle$, where

1. $\mathcal{M}_p = \langle M_i^p : i < n_p \rangle$ is a finite $\in$-increasing sequence of elements of $\mathcal{S}$, and
2. the function $f_p : \mathcal{M}_p \longrightarrow H_{\omega_1}$ is defined such that $f_p(M_i^p)$ is a finite subset of M_{i+1}^p if $i < n_p - 1$, and $f_p(M_{n_p-1}^p)$ is a finite subset of H_{ω_1}.

$q \leq p$ if and only if $\mathcal{M}_p \subseteq \mathcal{M}_q$ and $f_p(M) \subseteq f_q(M)$ for every $M \in \mathcal{M}_p$.

Lemma

For every $p \in \mathbb{P}$ *and* $\gamma \in \omega_1$, *there is* $p' \leq p$ *such that* $\gamma < \delta_N$ *for some* $N \in \mathcal{M}_{p'}$.

Lemma

For every $p \in \mathbb{P}$ and $\gamma \in \omega_1$, there is $p' \leq p$ such that $\gamma < \delta_N$ for some $N \in \mathcal{M}_{p'}$.

Proof

There exists $N \prec H_{\omega_1}$ such that $p, \gamma \in N$. Let $p' = \langle \mathcal{M}_{p'}, f_{p'} \rangle$ be such that:

- $\mathcal{M}_{p'} = \mathcal{M}_p \cup \{N\}$
- $f_{p'}(M) = f_p(M) \qquad M \in \mathcal{M}_p$
- $f_{p'}(N) = \emptyset$

Lemma

For every $p \in \mathbb{P}$ *and* $\gamma \in \omega_1$, *there is* $p' \leq p$ *such that* $\gamma < \delta_N$ *for some* $N \in \mathcal{M}_{p'}$.

Proof

There exists $N \prec H_{\omega_1}$ such that $p, \gamma \in N$. Let $p' = \langle \mathcal{M}_{p'}, f_{p'} \rangle$ be such that:

- $\mathcal{M}_{p'} = \mathcal{M}_p \cup \{N\}$
- $f_{p'}(M) = f_p(M) \qquad M \in \mathcal{M}_p$
- $f_{p'}(N) = \emptyset$

Lemma

If $G \subseteq \mathbb{P}$ *is a generic filter, then*

$$C = \{\delta_M : M \in \mathcal{M}_p \textit{ for some } p \in G\}$$

is a club of ω_1.

Given any $\gamma \in \omega_1$

Given any $\gamma \in \omega_1$

$D_\gamma = \{q \in \mathbb{P} \colon \exists M \in \mathcal{M}_q(\gamma < \delta_M)\}$ is dense

Given any $\gamma \in \omega_1$
$D_\gamma = \{q \in \mathbb{P} \colon \exists M \in \mathcal{M}_q (\gamma < \delta_M)\}$ is dense

C is unbounded in ω_1

Given any $\gamma \in \omega_1$
$D_\gamma = \{q \in \mathbb{P}\colon \exists M \in \mathcal{M}_q(\gamma < \delta_M)\}$ is dense

$$C \text{ is unbounded in } \omega_1$$

claim: If p forces γ to be a limit point of $\dot{C}$, then p also forces it is an element of $\dot{C}$.

Given any $\gamma \in \omega_1$
$D_\gamma = \{q \in \mathbb{P} \colon \exists M \in \mathcal{M}_q(\gamma < \delta_M)\}$ is dense

$$C \text{ is unbounded in } \omega_1$$

claim: If p forces γ to be a limit point of $\dot{C}$, then p also forces it is an element of $\dot{C}$.
Proof: Suppose not.

Given any $\gamma \in \omega_1$
$D_\gamma = \{q \in \mathbb{P} \colon \exists M \in \mathcal{M}_q(\gamma < \delta_M)\}$ is dense

$$C \text{ is unbounded in } \omega_1$$

claim: If p forces γ to be a limit point of $\dot{C}$, then p also forces it is an element of $\dot{C}$.
Proof: Suppose not. There is no $M \in \mathcal{M}_p$ with $\delta_M = \gamma$

Given any $\gamma \in \omega_1$
$D_\gamma = \{q \in \mathbb{P} \colon \exists M \in \mathcal{M}_q(\gamma < \delta_M)\}$ is dense

C is unbounded in ω_1

claim: If p forces γ to be a limit point of $\dot{C}$, then p also forces it is an element of $\dot{C}$.
Proof: Suppose not. There is no $M \in \mathcal{M}_p$ with $\delta_M = \gamma$
For some i, $\delta_{M_i^p} < \gamma < \delta_{M_{i+1}^p}$

Given any $\gamma \in \omega_1$
$D_\gamma = \{q \in \mathbb{P} \colon \exists M \in \mathcal{M}_q(\gamma < \delta_M)\}$ is dense

$$C \text{ is unbounded in } \omega_1$$

claim: If p forces γ to be a limit point of $\dot{C}$, then p also forces it is an element of $\dot{C}$.
Proof: Suppose not. There is no $M \in \mathcal{M}_p$ with $\delta_M = \gamma$
For some i, $\delta_{M_i^p} < \gamma < \delta_{M_{i+1}^p}$
Let $\xi < \delta_{M_{i+1}^p}$ be any ordinal greater than γ

Given any $\gamma \in \omega_1$
$D_\gamma = \{q \in \mathbb{P} \colon \exists M \in \mathcal{M}_q(\gamma < \delta_M)\}$ is dense

C is unbounded in ω_1

claim: If p forces γ to be a limit point of $\dot{C}$, then p also forces it is an element of $\dot{C}$.
Proof: Suppose not. There is no $M \in \mathcal{M}_p$ with $\delta_M = \gamma$
For some i, $\delta_{M_i^p} < \gamma < \delta_{M_{i+1}^p}$
Let $\xi < \delta_{M_{i+1}^p}$ be any ordinal greater than γ
$q = \langle \mathcal{M}_q, f_q \rangle$

- $\mathcal{M}_q = \mathcal{M}_p$
- $f_q(M) = f_p(M)$ for all $M \neq M_i^p$
- $f_q(M_i^p) = f_p(M_i^p) \cup \{\xi\}$

$\mathbb{P}$ is strongly proper.

$\mathbb{P}$ is not ω-proper.

Proof.

Let $p \in \mathbb{P} \cap N_0 \in \mathcal{N}$ an ω-tower

$\mathbb{P}$ is strongly proper.

$\mathbb{P}$ is not ω-proper.

Proof.

Let $p \in \mathbb{P} \cap N_0 \in \mathcal{N}$ an ω-tower

Let $q \leq p$ is $(\mathcal{N}, \mathbb{P})$-generic condition

$\mathbb{P}$ is strongly proper.

$\mathbb{P}$ is not ω-proper.

Proof.

Let $p \in \mathbb{P} \cap N_0 \in \mathcal{N}$ an ω-tower

Let $q \leq p$ is $(\mathcal{N}, \mathbb{P})$-generic condition

$q \Vdash$ "$\dot{C} \cap \delta_{N_i}$ is unbounded in δ_{N_i}", hence $q \Vdash$ "$\delta_{N_i} \in \dot{C}$".

There are n, i and $q' \leq q$, such that $\delta_{N_i^{q'}} < \delta_{N_n} < \delta_{N_\omega} < \delta_{N_{i+1}^{q'}}$.

$\mathbb{P}$ is strongly proper.

$\mathbb{P}$ is not ω-proper.

Proof.

Let $p \in \mathbb{P} \cap N_0 \in \mathcal{N}$ an ω-tower
Let $q \leq p$ is $(\mathcal{N}, \mathbb{P})$-generic condition
$q \Vdash$ "$\dot{C} \cap \delta_{N_i}$ is unbounded in δ_{N_i}", hence $q \Vdash$ "$\delta_{N_i} \in \dot{C}$".
There are n, i and $q' \leq q$, such that $\delta_{N_i^{q'}} < \delta_{N_n} < \delta_{N_\omega} < \delta_{N_{i+1}^{q'}}$.
Let $\xi < \delta_{N_{i+1}^{q'}}$ be such that $\delta_{N_\omega} < \xi$

$\mathbb{P}$ is strongly proper.

$\mathbb{P}$ is not ω-proper.

Proof.

Let $p \in \mathbb{P} \cap N_0 \in \mathcal{N}$ an ω-tower

Let $q \leq p$ is $(\mathcal{N}, \mathbb{P})$-generic condition

$q \Vdash$ "$\dot{C} \cap \delta_{N_i}$ is unbounded in δ_{N_i}", hence $q \Vdash$ "$\delta_{N_i} \in \dot{C}$".

There are n, i and $q' \leq q$, such that $\delta_{N_i^{q'}} < \delta_{N_n} < \delta_{N_\omega} < \delta_{N_{i+1}^{q'}}$.

Let $\xi < \delta_{N_{i+1}^{q'}}$ be such that $\delta_{N_\omega} < \xi$

let q'' be such that

- $\mathcal{M}_{q''} = \mathcal{M}_{q'}$
- $f_{q''}(M) = f_{q'}(M)$ for all $M \neq N_i^{q'}$
- $f_{q''}(N_i^{q'}) = f_{q'}(N_i^{q'}) \cup \{\xi\}$

$\mathbb{P}$ is strongly proper.

$\mathbb{P}$ is not ω-proper.

Proof.

Let $p \in \mathbb{P} \cap N_0 \in \mathcal{N}$ an ω-tower

Let $q \leq p$ is $(\mathcal{N}, \mathbb{P})$-generic condition

$q \Vdash$ "$\dot{C} \cap \delta_{N_i}$ is unbounded in δ_{N_i}", hence $q \Vdash$ "$\delta_{N_i} \in \dot{C}$".

There are n, i and $q' \leq q$, such that $\delta_{N_i^{q'}} < \delta_{N_n} < \delta_{N_\omega} < \delta_{N_{i+1}^{q'}}$.

Let $\xi < \delta_{N_{i+1}^{q'}}$ be such that $\delta_{N_\omega} < \xi$

let q'' be such that

- $\mathcal{M}_{q''} = \mathcal{M}_{q'}$
- $f_{q''}(M) = f_{q'}(M)$ for all $M \neq N_i^{q'}$
- $f_{q''}(N_i^{q'}) = f_{q'}(N_i^{q'}) \cup \{\xi\}$

$q'' \leq q$ such that for all $r \leq q''(r \Vdash$ "$\delta_{N_n} \notin \dot{C}$"). $\square$

Definition

Let $\alpha < \omega_1$ be an indecomposable ordinal.
$p = \langle \mathcal{M}_p, f_p, \mathcal{W}_p \rangle \in \mathbb{P}[\alpha]$ such that:

- $\mathcal{M}_p = \langle M^p_\xi \colon \xi \leq \gamma_p \rangle$ is an $\in$-increasing sequence of elements of $\mathcal{S}$ for some $\gamma_p < \alpha$ which is continuous at limits;
- the function $f_p : \mathcal{M}_p \longrightarrow H_{\omega_1}$ is defined such that $f_p(M^p_\xi)$ is a finite subset of $M^p_{\xi+1}$ for $\xi < \gamma$, and $f_p(M^p_\gamma)$ is a finite subset of H_{ω_1}; and
- the witness $\mathcal{W}_p$ is a countable $\in$-chain of elements of $\mathcal{S}$ such that for every $N \in \mathcal{W}_p$,
  $p\restriction_N = \langle \mathcal{M}_p \cap N, f_p\restriction_{\mathcal{M}_p \cap N}, \mathcal{W}_p \cap N \rangle \in N$.

$q \leq p$ if and only if $\mathcal{M}_p \subseteq \mathcal{M}_q$, $f_p(M) \subseteq f_q(M)$ for every $M \in \mathcal{M}_p$ and $\mathcal{W}_p \subseteq \mathcal{W}_q$.

Lemma

If $G \subseteq \mathbb{P}[\alpha]$ is a generic filter, then $C = \{\delta_M \colon M \in \mathcal{M}_p \text{ for some } p \in G\}$ is a club.

Theorem

$\mathbb{P}[\alpha]$ is β-proper for every $\beta < \alpha$, but is not α-proper.

Thank You