Adding Highly Generic Subsets of ω_2

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1401/12/3

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Gödel 1939: $Con(ZF)$ *implies* $Con(ZFC + GCH)$

Cohen 1963: $Con(ZF)$ *implies* $Con(ZF + \neg AC)$
$Con(ZFC)$ *implies* $Con(ZFC + \neg CH)$

$$V[G] \models ZFC + 2^{\aleph_0} \geq \aleph_2$$

FORCING

Forcing Notion

$\mathbb{P} = \langle P, \leq_P \rangle \in V$ a poset of **conditions** with largest element

$$Add(\aleph_0, \aleph_2) = \{p \vdots (\omega \times \omega_2) \longrightarrow 2 \mid |p| < \aleph_0\}$$

$p_1 \leq p_2$ **or** p_1 extends p_2 **or** p_1 is stronger than p_2 **or** p_1 has more information than p_2 **iff** $p_2 \subseteq p_1$

Dense Open subsets

$D \subseteq \mathbb{P}$ is dense open if

- $\forall p \in \mathbb{P} \exists d \in D (d \leq p)$
- $d_1 \in D \wedge d_2 \leq d_1 \Rightarrow d_2 \in D$

Generic Set

G is Generic if

- $G \subseteq \mathbb{P}$ is a filter
- G meets every dense open subset of $\mathbb{P}$ that lies in V

PRESERVING CARDINALS AND THE GCH

Cohen showed that $(2^{\aleph_0})^{V[G]} \geq \aleph_2^V$
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Chain condition

$\mathbb{P}$ is $\kappa.c.c.$ if every maximal antichain $A \subseteq \mathbb{P}$ has size $< \kappa$

Cohen: If $\mathbb{P}$ is $\kappa.c.c.$ then forcing with $\mathbb{P}$ preserves cardinals $\geq \kappa$.

$\aleph_1$-PRESERVING FORCINGS

Recall: Let X be a set and $\kappa \leq |X|$ a cardinal

$$[X]^\kappa = \{Y \subseteq X \mid |Y| = \kappa\}$$

Proper Forcings

A forcing notion $\mathbb{P}$ is proper if for every infinite X and every stationary set $S \subseteq [X]^{\leq \aleph_0}$, S remains stationary in $V[G]$

- **Shelah: If $\mathbb{P}$ is proper then forcing with $\mathbb{P}$ preserves $\aleph_1$**

Let θ be a regular uncountable cardinal

- $H(\theta) = \{x : |TC(x)| < \theta\}$
  $= \{x : x \subseteq y \ \exists y(y \text{ is transitive} \wedge |y| < \theta)\}$
- $\langle H(\theta), \in\rangle \models ZFC - P$
- If θ is inaccessible cardinal then $\langle H(\theta), \in\rangle \models ZFC$

Elementary substructures

$\mathfrak{A} \prec \mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$ and for all formulas $\varphi[x_1, \dots, x_n]$ of $\mathscr{L}$ and all $a_1, \dots, a_n \in A$, we have

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n] \iff \mathfrak{B} \models \varphi[a_1, \dots, a_n]$$

Let $\mathfrak{B}$ be a model of power α, let $|\mathscr{L}| \leq \beta \leq \alpha$, let $X \subseteq B$ and $|X| \leq \beta$ Then $\mathfrak{B}$ has an elementary submodel of power β which contains X

FORCING WITH ELEMENTARY SUBSTRUCTURES

$$\mathcal{S} = \{M \in [H(\theta)]^{\aleph_0} : \langle M, \in, <_w \rangle \prec \langle H(\theta), \in, <_w \rangle\}$$

THE $\in$-COLLAPSE FORCING

$\mathbb{P}_\in(\theta)$ is the set of all finite $\in$-chains of countable elementary submodels of $\langle H(\theta), \in, \leq_w \rangle$ with the inverse inclusion as the order

MATRIX $\in$-COLLAPSE FORCING

$\mathbb{P}_\in^{\mathcal{M}} = \{p \subset \mathcal{S} \mid |p| < \aleph_0\}$ such that

- If $M, N \in p$ and $M \cap \omega_1 = \delta_M = \delta_N = N \cap \omega_1$, then $\langle M, \in, <_w \rangle \simeq \langle N, \in, <_w \rangle$
- If $M \in p$ and $\delta \in dom(p)$ such that $\delta_M < \delta$, then $\exists N \in p(\delta)\ (M \in N)$.
  where $p(\alpha) = \{M \in p : \delta_M = \alpha\}$ and $dom(p) = \{\alpha : p(\alpha) \neq \emptyset\}$.

GITIK'S QUESTION (2017)

Suppose GCH holds and κ is a regular cardinal. Is there a cardinal and GCH preserving extension of the universe in which there exists a set $A \subseteq \kappa$ of size κ such that for all countable set $X \in \mathscr{P}(\kappa) \cap V$, $A \cap X$ and $X \backslash A$ are non-empty?

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$\kappa = \aleph_0$

$\mathbb{P}_\omega = \{p : \omega \longrightarrow 2 \mid |p| < \aleph_0 \}$

$\kappa = \aleph_1$

$\mathbb{P}_{\omega_1} = \{p : \omega_1 \longrightarrow 2 \mid |p| < \aleph_0 \}$

DENSITY ARGUMENT

$D_X = \{p \in \mathbb{P} : \exists\alpha, \beta \in X \cap dom(p)\big(p(\alpha) = 1 \wedge p(\beta) = 0\big)\}$.

If $p \in G \cap D_X$ and $\alpha, \beta \in X \cap dom(p)$ be such that $p(\alpha) = 1$ and $p(\beta) = 0$, then $\alpha \in X \cap A$ and $\beta \in X \setminus A$

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THE PROBLEM

$\mathbb{P}_{\omega_2} \simeq Add(\aleph_0, \aleph_2)$ Hence $2^{\aleph_0} > \aleph_1$ and GCH fails

THE FORCING NOTION

Definition

A condition p of $\mathbb{P}$ is a pair $\langle \mathcal{M}_p, f_p \rangle$, whenever:

(i) $\mathcal{M}_p \in \mathbb{P}^M_\in$;

(ii) $f_p : \omega_2 \longrightarrow 2$ is a finite partial function;

(iii) If $M, N \in \mathcal{M}_p$ with $\delta_M = \delta_N$, then
- $\alpha \in (dom(f_p) \cap M) \Rightarrow \varphi_{M,N}(\alpha) \in dom(f_p)$,
- for each α as above, $f_p(\varphi_{M,N}(\alpha)) = f_p(\alpha)$.

For $p, q \in \mathbb{P}$, we say $p \leq q$ if and only if $\mathcal{M}_q \subseteq \mathcal{M}_p$ and $f_q \subseteq f_p$.

Lemma

$\mathbb{P}$ is strongly proper and satisfies the $\aleph_2$-c.c

Lemma

Forcing with $\mathbb{P}$ preserves the CH.

Thank You