Forcing With Elementary Substructures

(Side condition forcing)

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Gödel 1939: $Con(ZF)$ *implies* $Con(ZFC + GCH)$

Cohen 1963: $Con(ZF)$ *implies* $Con(ZF + \neg AC)$
$Con(ZFC)$ *implies* $Con(ZFC + \neg CH)$

$$V[G] \models ZFC + 2^{\aleph_0} \geq \aleph_2$$

Forcing Notion

$\mathbb{P} = \langle P, \leq_P \rangle \in V$ a poset of **conditions** with largest element

$$Add(\aleph_0, \aleph_2) = \{p \vdots (\omega \times \omega_2) \longrightarrow 2 \mid |p| < \aleph_0\}$$

$p_1 \leq p_2$ **or** p_1 extends p_2 **or** p_1 is stronger than p_2 **or** p_1 has more information than p_2 **iff** $p_2 \subseteq p_1$

Dense Open subsets

$D \subseteq \mathbb{P}$ is dense open if

- $\forall p \in \mathbb{P} \exists d \in D (d \leq p)$
- $d_1 \in D \wedge d_2 \leq d_1 \Rightarrow d_2 \in D$

Generic Set

G is Generic if

- $G \subseteq \mathbb{P}$ is a filter
- G meets every dense open subset of $\mathbb{P}$ that lies in V

$V^{\mathbb{P}}$ the class of $\mathbb{P}$-names

$\tau \in V^{\mathbb{P}} \iff \tau$ is a relation and $\forall \langle \sigma, p \rangle \in \tau (\sigma \in V^{\mathbb{P}} \wedge p \in \mathbb{P})$

- $V^{\mathbb{P}} \subset V$
- $G \notin V$

$V[G]$ the generic extension

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- $V[G]$ is the minimal **model** that includes $V \cup \{G\}$ and $V \cap Ord = V[G] \cap Ord$
- $\forall \varphi (V[G] \models \varphi \iff \exists p \in G (p \Vdash \varphi))$

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Chain condition

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Cohen: If $\mathbb{P}$ is $\kappa.c.c.$ then forcing with $\mathbb{P}$ preserves cardinals $\geq \kappa$.

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κ-closed

$\mathbb{P}$ is κ-closed if for every $\lambda < \kappa$, every descending sequence $\langle p_\alpha : \alpha < \lambda \rangle$ of conditions of $\mathbb{P}$ has a lower bound.
Solovay: If $\mathbb{P}$ is κ-closed then forcing with $\mathbb{P}$ preserves cardinals $\leq \kappa$

Axiom A Forcings

A class of $\aleph_1$ preserving forcing notions that contains $\aleph_1.c.c.$ and $\aleph_1$-closed forcing notions introduced by Baumgartner

Recall: Let X be a set and $\kappa \leq |X|$ a cardinal

$$[X]^{\kappa} = \{\, Y \subseteq X \mid |Y| = \kappa \}$$

Proper Forcings

A forcing notion $\mathbb{P}$ is proper if for every infinite X and every stationary set $S \subseteq [X]^{\leq \aleph_0}$, S remains stationary in $V[G]$

- **Shelah: If $\mathbb{P}$ is proper then forcing with $\mathbb{P}$ preserves $\aleph_1$**
- Axiom A forcings are proper

Let θ be a regular uncountable cardinal

- $H(\theta) = \{x : \ |TC(x)| < \theta\}$
  $= \{x : \ x \subseteq y \ \exists y(y \text{ is transitive} \wedge |y| < \theta)\}$
- $\langle H(\theta), \in\rangle \models ZFC - P$
- If θ is inaccessible cardinal then $\langle H(\theta), \in\rangle \models ZFC$

Elementary substructures

$\mathfrak{A} \prec \mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$ and for all formulas $\varphi[x_1, \ldots, x_n]$ of $\mathscr{L}$ and all $a_1, \ldots, a_n \in A$, we have

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_n] \iff \mathfrak{B} \models \varphi[a_1, \ldots, a_n]$$

Let $\mathfrak{B}$ be a model of power α, let $|\mathscr{L}| \leq \beta \leq \alpha$, let $X \subseteq B$ and $|X| \leq \beta$ Then $\mathfrak{B}$ has an elementary submodel of power β which contains X

Let θ be a large enough regular cardinal, $\mathbb{P}$ a forcing notion and $\langle N, \in, \leq_w \rangle \prec \langle H(\theta), \in, \leq_w \rangle$ countable with $\mathbb{P} \in N$

Generic Conditions

$q \in \mathbb{P}$ is an $(N, \mathbb{P})$-generic condition if for every dense open set $D \subseteq \mathbb{P}$, $D \in N$ implies $D \cap N$ is predense below q. i.e.

$$q' \leq q \longrightarrow \exists d \in D \cap N \, \exists p \in \mathbb{P}(p \leq q', d)$$

$\mathbb{P}$ is proper iff for all such N, every $p \in \mathbb{P} \cap N$ has an $(N, \mathbb{P})$-generic extension

$q \in \mathbb{P}$ is $(N, \mathbb{P})$-strongly generic condition if every dense open set $D \subseteq \mathbb{P} \cap N$ is predense below q

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Let $\kappa > \theta$ be a large enough regular cardinal and $\langle M', \in, <_w\rangle \prec \langle H(\kappa), \in, <_w\rangle$ with $\theta \in M'$ and $M = M' \cap H(\theta)$

- If $M \in q$, then $q \cap M' \in \mathbb{P}_{\in}(\theta) \cap M'$
- If $p \in \mathbb{P}_{\in}(\theta) \cap M'$, then $p \cup \{M\} \in \mathbb{P}_{\in}(\theta)$.

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PFA

For every proper forcing $\mathbb{P}$ and for every family $\{D_\alpha : \alpha \in \omega_1\}$ of dense sets of $\mathbb{P}$, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha \in \omega_1$

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Applications

PFA implies *OGA*, *PID*, $BA(\omega_1)$

$\mathbb{P} = \{p : \alpha \longrightarrow \omega_1 \mid \alpha \subset \omega_1 \text{is finite}\}$ with inverse inclusion

- $\mathbb{P}$ is not $\aleph_1.c.c.$
- It collapses $\aleph_1$
- we can not prove its properness

Let $M \cap \omega_1 = \delta$, $\alpha \in M \backslash dom(p)$ and $q(a) > \delta$, then the natural restriction of q to M as a function is not in M

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Side conditions

$\mathbb{P}' = \{(p, \mathcal{N}) : \ p \in \mathbb{P} \wedge \mathcal{N} \in \mathbb{P}_{\in}(\theta)\}$ such that

$$\forall N \in \mathcal{N} \ \forall \alpha \in dom(p) \cap N \ (p(\alpha) \in N)$$

Gitik's question (2017)

Suppose GCH holds and κ is a regular cardinal. Is there a cardinal and GCH preserving extension of the universe in which there exists a set $A \subseteq \kappa$ of size κ such that for all countable set $X \in \mathscr{P}(\kappa) \cap V$, $A \cap X$ and $X \backslash A$ are non-empty?

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$\kappa = \aleph_0$

$\mathbb{P}_\omega = \{p : \omega \longrightarrow 2 \mid |p| < \aleph_0 \}$

$\kappa = \aleph_1$

$\mathbb{P}_{\omega_1} = \{p : \omega_1 \longrightarrow 2 \mid |p| < \aleph_0 \}$

$\mathbb{P}_{\omega_2} \simeq Add(\aleph_0, \aleph_2)$ Hence $2^{\aleph_0} > \aleph_1$ and GCH fails

$$\mathcal{S} = \{M \in [H(\theta)]^{\aleph_0} : \langle M, \in, <_w\rangle \prec \langle H(\theta), \in, <_w\rangle\}$$

$\mathbb{P}_{\in}^{\mathcal{M}} = \{p \subset \mathcal{S} \mid |p| < \aleph_0\}$ such that

- If $M, N \in p$ and $M \cap \omega_1 = \delta_M = \delta_N = N \cap \omega_1$, then $\langle M, \in, <_w\rangle \simeq \langle N, \in, <_w\rangle$
- If $M \in p$ and $\delta \in dom(p)$ such that $\delta_M < \delta$, then $\exists N \in p(\delta)\ (M \in N)$.
  where $p(\alpha) = \{M \in p : \delta_M = \alpha\}$ and $dom(p) = \{\alpha : p(\alpha) \neq \emptyset\}$.

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- If $M \in p$ and $\delta \in dom(p)$ such that $\delta_M < \delta$, then $\exists N \in p(\delta)\ (M \in N)$.
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Aplication

- $\mathbb{P}_{\in}^{\mathcal{M}}$ is strongly proper and forces the Continuum Hypothesis
- In $V[G]$ there is a Kurepa tree with exactly ω_2 branches that does not contain Aronszajn subtrees
- (We solved) the Gitik's question for $\kappa = \aleph_2$

$\kappa < \lambda < \theta$

- $\mathcal{T}$ is a collection of transitive $\langle W, \in, <_w \rangle \prec \langle H(\theta), \in, <_w \rangle$, and $\mathcal{S}$ is a collection of $\langle M, \in, <_w \rangle \prec \langle H(\theta), \in, <_w \rangle$ with $\kappa \subset M$ and $|M| < \lambda$. All elements of $\mathcal{S} \cup \mathcal{T}$ belong to $H(\theta)$ and contain $\{\kappa, \lambda\}$
- If $M_1, M_2 \in \mathcal{S}$ and $M_1 \in M_2$ then $M_1 \subset M_2$
- If $M \in \mathcal{S}$ and $W \in M \cap \mathcal{T}$ then $(M \cap W) \in W \cap \mathcal{S}$
- Each $W \in \mathcal{T}$ is closed under sequences of lenght $\leq \kappa$ in $H(\theta)$

$\mathbb{P}_{\kappa,\lambda}^{\mathcal{S},\mathcal{T}} = \{\langle M_\xi : \xi < \gamma \rangle \in H(\theta) \mid \gamma < \kappa\}$ such that

- $\forall \xi$, $M_\xi \in \mathcal{S} \cup \mathcal{T}$
- $\forall \zeta < \gamma$, $\{\xi < \zeta : M_\xi \in M_\zeta\}$ is cofinal in ζ
- $\forall \zeta < \gamma$, $\langle M_\xi : \xi < \zeta \wedge M_\xi \in M_\zeta \rangle \in M_\zeta$
- The sequence is closed under intersections

$$\mathcal{S} = \{M \in [H(\theta)]^{\aleph_0} : \langle M, \in, <_w\rangle \prec \langle H(\theta), \in, <_w\rangle\}$$
$$\mathcal{T} = \{H(\lambda) : \langle H(\lambda), \in, <_w\rangle \prec \langle H(\theta), \in, <_w\rangle \wedge cf(\lambda) > \omega\}$$

1

$\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$ preserves $\aleph_1$
If λ be a cardinal such that $\omega_1 < \lambda < \theta$, collapsed to ω_1
If $\mathcal{T}$ is stationary set on $[H(\theta)]^{<\theta}$ then $\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}} \Vdash \theta = \omega_2$

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2

Let θ a Supercompact cardinal and $J : \theta \longrightarrow H(\theta)$ Laver function
$\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}(J)$ is an iterated forcing, constructed using $\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$
preserves $\aleph_1$
forces that $\theta = \aleph_2$
Get new proof for the consistency of PFA
$V[G]$ is a model of PFA in which PFA^+ fails

The Problem

The forcing of Neeman preserves cardinals but does not preserve GCH

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Replacing the linear parts of small nodes with matrices of the same type of models

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Let

$$\mathcal{S} = \{M \in [H(\omega_2)]^{\aleph_0} : \langle M, \in, <_w\rangle \prec \langle H(\omega_2), \in, <_w\rangle\}$$

$$\mathcal{T} = \{X \in [H(\omega_2)]^{\aleph_1} : \langle X, \in, <_w\rangle \prec \langle H(\omega_2), \in, <_w\rangle \wedge {}^{\omega}X \subset X\}$$

- Is there a cardinal and GCH preserving forcing notion $\mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in,\mathcal{M}}$?
- Can we iterete such a forcing preserving GCH?

REFRENCES (Papers by date)

- Todorčević S., Kuzeljevic and Borisa, Forcing with matrices of countable elementary submodels, Proceedings of the American Mathematical Society 145(5), 2211-2222, 2017
- Krueger J., Forcing with adequate sets of models as side conditions, Mathematical Logic Quarterly 63(1), 124-149, 2017
- Neeman I., Forcing with sequences of models of two types, Notre Dame Journal of Formal Logic 55(2), 265-298, 2014
- Krueger J., Strongly adequate sets and adding a club with finite conditions, Archive for Mathematical Logic 53(1), 119-136, 2014
- Mitchell W. J., Adding Closed Unbounded Subsets of ω_2 with Finite Forcing, Notre Dame Journal of Formal Logic 46(3), 357-371, 2005

REFRENCES (Papers by date)

- Koszmider P., Models as side conditions, Set theory, Springer, Dordrecht, 99-107, 1998
- Todorčević S., Partition relations for partially ordered sets, Acta Mathematica 155, 1-25, 1985
- Todorčević S., Directed sets and cofinal types, Transactions of the American Mathematical Society 290(2), 711-723, 1985
- Todorčević S., A note on the proper forcing axiom, Contemporary Mathematics 95, 209-218, 1984
- Baumgartner J., Iterated forcing, Surveys in set theory, 1-59, 1983
- Todorčević S., Guzmán and Osvaldo, The P-Ideal Dichotomy, Martin's Axiom and Entangled Sets

REFRENCES (Books by date)

- Shelah S., Proper and improper forcing, 2nd ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.
- Kunen K., Set theory an introduction to independence proofs, Elsevier, 2014
- Foreman M. and Kanamori A., Handbook of Set Theory, Germany, Springer Netherlands, 2010
- Jech T., Set theory: The Third Millennium Edition, Springer, 2003
- Chang C. C., and Keisler H. J., Model theory, Elsevier, 1990
- Todorčević S., Partition problems in topology, Vol. 84, American Mathematical Soc., 1989

Thank You